

An integrable model with parafermion zero energy modes

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Parafermion zero energy modes are a vital element of fault-tolerant topological quantum computation. Although it is believed that such modes form on the border between topological and normal phases, this has been demonstrated only for Z_2 (Majorana) and Z_3 parafermions. I consider an integrable model where such demonstration is possible for Z_N parafermions with any N . The procedure is easily generalizable for more complicated symmetry groups.

Non-Abelian anyons possess the most exotic statistics known to man. Their permutation transforms one ground state into another one locally indistinguishable from the first [1],[2]. In conformal field theories this property appears as non-trivial braiding of the conformal blocks [3] (see also [4]). There are possible applications of non-Abelian statistics to fault-tolerant topological quantum computation [5],[6],[7],[8], but it is also interesting on its own right.

The simplest anyons emerge in models of Majorana fermions which material realization has possibly been already achieved [9]. The conceptually simplest and most straightforward generalization of Majorana fermions is Z_N parafermions. The former ones have Z_2 symmetry and the parafermions have Z_N ($N>2$) symmetry. In quantum computation applications one needs to manipulate parafermionic zero energy modes for which end several schemes have been recently suggested [10],[11]. In systems with many anyon zero modes they will unavoidably interact so that the degeneracy will be lifted. This is also an interesting subject of research and many lattice models of interacting anyons have been considered (see, for example [12],[13],[14] and references therein).

The problem number one in this business is how to obtain anyon zero energy modes. It has been argued, in direct analogy with the Majorana zero modes, that they emerge on a boundary between ground states with different topological properties (see, for example, [11]). The problem is, however, that for $N > 2$ the parafermions are interacting objects which makes a consideration of inhomogeneous cases difficult. So far the existence of the zero modes was demonstrated only for $N=3$ case which can be treated by the Abelian bosonization [11].

Here I suggest a solvable model which contains an inhomogeneity of the required type. The analysis of the corresponding Bethe ansatz equations supports the idea that a boundary between topologically different states does contain parafermionic zero modes. Although it is not a proof that parafermion zero modes *always* emerge on the boundary between ground states of different topology, but this is at least a demonstration that they may emerge there. The derivation is easily generalizable to parafermions from other simple Lie groups such as, for instance, $SU_k(N)$ (see, for example, [15]). I also derive an effective model describing a finite density of such modes and obtain its exact solution. The latter solutions allows one to estimate the interaction strength between the parafermions.

The field theoretical definition and properties of massless Z_N chiral parafermionic fields ψ, ψ^+ and $\bar{\psi}, \bar{\psi}^+$ can be extracted from the $SU_N(2)$ Kac-Moody algebra. The corresponding current operators can be defined in terms of free chiral fermion fields R, R^+ and L, L^+ :

$$J^a = \sum_{k=1}^N R_{k\alpha}^+ S_{\alpha\beta}^a R_{k\beta}, \quad \bar{J}^a = \sum_{k=1}^N L_{k\alpha}^+ S_{\alpha\beta}^a L_{k\beta}, \quad (1)$$

where S^a are spin $S=1/2$ matrices. On the other hand these currents can be written as [16]

$$J^+ = \frac{\sqrt{N}}{2\pi} e^{i\sqrt{8\pi/N}\varphi} \psi, \quad J^- = \frac{\sqrt{N}}{2\pi} e^{-i\sqrt{8\pi/N}\varphi} \bar{\psi}^+ \quad (2)$$

$$\begin{aligned} J^z &= i\sqrt{N/2\pi} \partial_z \varphi \\ \bar{J}^+ &= e^{-i\sqrt{8\pi/N}\bar{\varphi}} \bar{\psi}^+, \quad \bar{J}^- = e^{i\sqrt{8\pi/N}\bar{\varphi}} \bar{\psi} \\ \bar{J}^z &= -i\sqrt{N/2\pi} \partial_{\bar{z}} \bar{\varphi}, \end{aligned} \quad (3)$$

where ψ, ψ^+ are chiral parafermion fields and $\varphi, \bar{\varphi}$ are chiral components of the bosonic scalar field $\Phi = \varphi + \bar{\varphi}$ governed by the Gaussian action

$$S = \frac{1}{2} \int d^2x (\partial_\mu \Phi)^2. \quad (4)$$

From (2,3) we deduce the expressions for the two-point correlators of the parafermion fields

$$\langle\langle \psi(z) \psi^+(0) \rangle\rangle \sim z^{-2(1-1/N)}, \quad \langle\langle \bar{\psi}(\bar{z}) \bar{\psi}^+(0) \rangle\rangle \sim \bar{z}^{-2(1-1/N)}. \quad (5)$$

which for $N > 2$ reveal their nontrivial braiding properties.

In a direct analog to the Majorana fermions one can introduce a mass term for the Z_N parafermions. The corresponding action is

$$S = Z_N[\psi, \bar{\psi}] - \lambda \int d^2x [\psi \bar{\psi} + \psi^+ \bar{\psi}^+], \quad (6)$$

where Z_N term describes the critical part of the parafermion action. For $N > 2$ this is an interacting theory though its properties can be studied since it is integrable[17].

In the $N=2$ case it is easy to study a situation where λ is coordinate dependent. It is well known that when $\lambda(x)$ changes sign (a kink) the Schrödinger equation has a zero energy solution where the eigenfunction is localized at the kink (zero energy Majorana bound state). An important question is whether such bound states exist for $N>2$. Here I suggest an indirect approach demonstrating existence of the parafermion zero modes.

Let us consider the $SU_N(2)$ Wess-Zumino-Novikov-Witten model perturbed by the anisotropic current-current interaction. The Hamiltonian density is

$$\mathcal{H} = \frac{2\pi}{N+2} [: J^a J^a : + : \bar{J}^a \bar{J}^a :] + g_{\parallel} J^z \bar{J}^z + g_{\perp} [J^+ \bar{J}^- + J^- \bar{J}^+] \quad (7)$$

The Hamiltonian (7) describes the $SU(2)$ -invariant subsector of the fermionic model with $U(1) \times U(1) \times SU(N)$ symmetry. This fermionic model was solved by the Bethe ansatz [19]. At $g_{\parallel} > 0$ the theory is massive and has solitons and antisolitons. As is evident from the exact solution, each (anti)soliton carries parafermion zero mode which supplies it with the non-Abelian statistics. The corresponding S-matrix in the soliton sector is a tensor product of the XXZ S-matrix (the scattering matrix of the sine-Gordon model) and the RSOS (Restricted Solid on Solid) one [20]. At sufficiently large g_{\parallel} it also has soliton-antisoliton bound states.

The Lagrangian density for Hamiltonian (7) can be written as

$$\mathcal{L} = \frac{1}{2}(\partial_{\mu}\Phi)^2 + Z_N[\psi, \bar{\psi}] - \lambda \left(e^{i\beta\Phi} \psi \bar{\psi} + H.c. \right), \quad (8)$$

where $\lambda \sim Ng_{\perp}$ and β is related to g_{\parallel} so that at small couplings we have $\beta^2 = (1 + Ng_z/4\pi)^{-1}$.

The last term in (8) is similar to the last term in (6) where the role of static function $\lambda(x)$ is played by dynamic field $\exp[i\beta\Phi]$. Since this field changes sign on soliton configuration one can use model (8) as a substitute for the model of parafermions (6) with a coordinate dependent mass gap provided one meets certain requirements. First, the solitons must be slow and on average be far from each other. Second, quantum fluctuations of the bosonic exponent should be small which requires small β .

These requirements are met in the following set up. Let us apply a magnetic field (it is coupled to the bosonic sector) which strength is slightly below the soliton mass threshold. The field breaks the symmetry between the solitons and the antisolitons. The magnitude of the field is slightly below the soliton mass so that

$$T \ll M - H \ll M. \quad (9)$$

At that temperatures we have a rarified gas of thermally excited slow solitons and no antisolitons. By looking at the thermodynamic Bethe ansatz (TBA) equations we can establish whether solitons carry parafermionic zero modes.

The corresponding TBA describing the soliton sector of the theory in the limit (9) can be extracted, for example, from [18]. They are a part of a more general system of equations which may contain also massive soliton-antisoliton bound states (see [21]) which are irrelevant for the present discussion.

$$F/L = -TM \int \frac{d\theta}{2\pi} \cosh \theta \ln(1 + e^{\epsilon_N(\theta)/T}), \quad (10)$$

$$\epsilon_j = Ts * \ln(1 + e^{\epsilon_{j-1}/T})(1 + e^{\epsilon_{j+1}/T}) + Ts * \ln(1 + e^{\epsilon_N/T}) \delta_{j,N-1}, \quad j = 1, \dots, N-1, \quad (11)$$

$$\epsilon_N - K * T \ln(1 + e^{\epsilon_N/T}) = -M \cosh \theta + H + Ts * \ln(1 + e^{\epsilon_{N-1}/T}) + O(e^{-H/T}). \quad (12)$$

where L is the system size and kernel K is

$$K(\omega) = \frac{\sinh[\pi(\xi - 1)\omega/2]}{2 \cosh(\pi\omega/2) \sinh(\pi\xi\omega/2)}, \quad \xi = \frac{1}{8\pi/N\beta^2 - 1}. \quad (13)$$

$$s * f(x) = \int_{-\infty}^{\infty} \frac{dy f(y)}{\pi \cosh(x - y)}$$

I am interested in limit (9). In the first approximation one can replace quasi-energies ϵ_j ($j=1,\dots,N-1$) by their constant asymptotic values for which the corresponding integral equations (11) become algebraic. The solution is

$$1 + e^{\epsilon_j/T} = \left\{ \frac{\sin \left[\frac{\pi(j+1)}{N+2} \right]}{\sin \left(\frac{\pi}{N+2} \right)} \right\}^2. \quad (14)$$

Substituting this into (10) we obtain the following expression for the free energy:

$$F/L = -TQ \int \frac{dp}{2\pi} e^{-(M-H)/T - p^2/2MT} + O(\exp[-2(M-H)/T]), \quad (15)$$

$$Q = 2 \cos \left(\frac{\pi}{N+2} \right). \quad (16)$$

This expression describes the free energy of an ideal gas of particles of mass M with a chemical potential H . The prefactor Q indicates that the state of \mathcal{N} particles with given energy is degenerate so that in the thermodynamic limit the degeneracy is equal to $Q^{\mathcal{N}}$. This degeneracy obviously comes from the parafermionic zero modes bound to the solitons. The fact that Q is not integer is a direct indication that the operators describing zero modes attached to different kinks do not commute with each other. For $N=2$ we reproduce the known result $D(2)_{\mathcal{N}} = 2^{[N/2]}$ for the dimensionality of the Clifford algebra representation of \mathcal{N} gamma matrices. For $N=3$ the obtained dimensionality is the large \mathcal{N} asymptotic of Fibonacci numbers:

$$\phi = 2 \cos(\pi/5) = \frac{1 + \sqrt{5}}{2}, \quad D(3)_{\mathcal{N}} = [\phi^{\mathcal{N}} - (-\phi)^{-\mathcal{N}}]/\sqrt{5}. \quad (17)$$

Expression (15) is the first term in the expansion of the free energy in the soliton density and, as I have said, describes the ideal gas of anyons. One can move further and extract from (11) the equations for interacting anyon gas. The interactions lift the ground state degeneracy.

At lowest temperatures we invert the matrix kernel in (11) to get the equations in the form where the kernel acts on the term which vanish in $T=0$ limit:

$$T \ln(1 + e^{\epsilon_j/T}) - T \mathcal{A}_{jk} \ln(1 + e^{-\epsilon_k/T}) = \mathcal{A}_{j,N-1} * s * T \ln(1 + e^{\epsilon_N/T}), \quad (18)$$

where

$$\mathcal{A}_{jk}(\omega) = 2 \coth(\pi\omega/2) \frac{\sinh\{\pi[N - \max(j,k)]\omega/2\} \sinh\{\pi \min(j,k)\omega/2\}}{\sinh(N\pi\omega/2)}.$$

At temperatures $T \ll M$ the distribution function in the right hand side (r.h.s.) of (18) is very sharp and can be replaced by delta function:

$$\mathcal{A}_{j,N-1} * s * T \ln(1 + e^{\epsilon_N(\theta)/T}) \approx n_{sol}(T) \mathcal{A}_{j,N-1} * s(\theta), \quad (19)$$

where n_{sol} is the number of solitons. Then Eqs.(18) with such r.h.s. look like the TBA for the ferromagnetic XXZ model with n_{sol} sites and anisotropy $\gamma = \pi/N$ with an additional restriction forbidding solutions with rapidities shifted by $i\pi/2$. Such restricted equations describe the critical RSOS models [22],[23] with conformal charge $c = 2(N-1)/(N+2)$. The r.h.s. of Eqs.(19) is $\sim n_{sol}$ which means that the bandwidth of the excitations of the interacting anyon gas is proportional to the average distance between the solitons $\lambda \sim n_{sol}^{-1}$. This contradicts a naive expectation that this bandwidth is proportional to the overlap of the zero mode wave functions. Such overlap would be exponentially small in $M\lambda$.

In this paper I used the integrable model to describe a state where the mass term of Z_N parafermions alternates its sign. The analysis of the Bethe ansatz equations show that the parafermions create zero energy bound states attached to the domain walls (solitons). When the density domain walls is finite, the modes interact which lifts the ground state degeneracy. The characteristic bandwidth of the excitations of this interacting anyon gas is proportional to the density of domain walls.

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